$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

It[illegible]n properties are that it is linear in both of its arguments, $[\hat{A}_1 + \hat{A}_2, \hat{B}] = [\hat{A}_1, \hat{B}] + [\hat{A}_1, \hat{B}]$ and $[\hat{A}, \hat{B}_1 + \hat{B}_2] = [\hat{A}, \hat{B}_1] + [\hat{A}, \hat{B}_2]$, and that it is antisymmetric, $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$.

1. Another important general property is the Jacobi identity

$$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0. \tag{8}$$

   Prove it.

2. Show that

$$[\hat{A}, \hat{B}_1\hat{B}_2] = [\hat{A}, \hat{B}_1]\hat{B}_2 + \hat{B}_1[\hat{A}, \hat{B}_2], \tag{9}$$

   and that in general

$$[\hat{A}, \hat{C}_1\hat{C}_2\cdots\hat{C}_n] = [\hat{A}, \hat{C}_1]\hat{C}_2\cdots\hat{C}_n + \hat{C}_1[\hat{A}, \hat{C}_2]\hat{C}_3\cdots\hat{C}_n + \cdots + \hat{C}_1\cdots\hat{C}_{n-1}[\hat{A}, \hat{C}_n]. \tag{10}$$

   Tip: use induction.

3. Let us now consider the position $\hat{x}$ and momentum $\hat{p} = -\mathrm{i}\hbar\partial_x$ operators. Using the commutation relation $[\hat{p}, \hat{x}] = -\mathrm{i}\hbar$, show that

$$[\hat{p}, f(\hat{x})] = -\mathrm{i}\hbar f'(\hat{x}) \tag{11}$$

   where f is an analytic function so $f(\hat{x})$ is defined via Eq. (2). (Matrices are the simplest examples of operators

$$[\hat{A}, \hat{B}_1\hat{B}_2]$$

$$= \hat{A}\hat{B}_1\hat{B}_2 - \hat{B}_1\hat{B}_2\hat{A}$$

$$= \hat{A}\hat{B}_1\hat{B}_2 - \hat{B}_1\hat{A}\hat{B}_2 - \hat{B}_1\hat{B}_2\hat{A} + \hat{B}_1\hat{A}\hat{B}_2$$

$$= (\hat{A}\hat{B}_1 - \hat{B}_1\hat{A})\hat{B}_2 + \hat{B}_1(\hat{A}\hat{B}_2 - \hat{B}_2\hat{A})$$

$$= [\hat{A}, \hat{B}_1]\hat{B}_2 + \hat{B}_1[\hat{A}, \hat{B}_2]$$

a.

Induktionsanfang $n = 3$

$$\cancel{[[\hat{A}, \hat{C}_1\hat{C}_2\hat{C}_3]] = [}$$

$$[\hat{A}, \hat{C}_1]\hat{C}_2\hat{C}_3 + \hat{C}_1[\hat{A}, \hat{C}_2]\hat{C}_3 + \hat{C}_1\hat{C}_2[\hat{A}, \hat{C}_3]$$

$$= (\hat{A}\hat{C}_1 - \hat{C}_1\hat{A})\hat{C}_2\hat{C}_3 + \hat{C}_1(\hat{A}\hat{C}_2 - \hat{C}_2\hat{A})\hat{C}_3$$
$$+ \hat{C}_1\hat{C}_2(\hat{A}\hat{C}_3 - \hat{C}_3\hat{A})$$

$$= \hat{A}\hat{C}_1\hat{C}_2\hat{C}_3 - \hat{C}_1\cancel{\hat{A}}\hat{C}_2\hat{C}_3 + \hat{C}_1\cancel{\hat{A}\hat{C}_2}\hat{C}_3 - \hat{C}_1\cancel{\hat{C}_2}\hat{A}\hat{C}_3$$
$$+ \hat{C}_1\cancel{\hat{C}_2\hat{A}}\hat{C}_3 - \hat{C}_1\hat{C}_2\hat{C}_3\hat{A}$$

$$= \hat{A}\hat{C}_1\hat{C}_2\hat{C}_3 - \hat{C}_1\hat{C}_2\hat{C}_3\hat{A} = [\hat{A}, \hat{C}_1\hat{C}_2\hat{C}_3] \quad \checkmark$$

~~Es gelte~~ Sei $n \geq 3$

Es gelte (IV)

$$[\hat{A}, \hat{C}_1 \hat{C}_2 \dots \hat{C}_n]$$

$$= [\hat{A}, \hat{C}_1] \hat{C}_2 \dots \hat{C}_n + \hat{C}_1 [\hat{A}, \hat{C}_2] \hat{C}_3 \dots \hat{C}_n + \hat{C}_1 \dots \hat{C}_{n-1} [\hat{A}, \hat{C}_n]$$

Induktionsschritt

$$[\hat{A}, \overbrace{\hat{C}_1 \hat{C}_2 \dots \hat{C}_n}^{\hat{B}_1} \overbrace{\hat{C}_{n+1}}^{\hat{B}_2}]$$

$$= [\hat{A}, \hat{C}_1 \hat{C}_2 \dots \hat{C}_n] \hat{C}_{n+1} + \hat{C}_1 \hat{C}_2 \dots \hat{C}_n [\hat{A}, \hat{C}_{n+1}]$$

$$\overset{IV}{=} \left([\hat{A}, \hat{C}_1] \hat{C}_2 \dots \hat{C}_n + \hat{C}_1 [\hat{A}, \hat{C}_2] \hat{C}_3 \dots \hat{C}_n + \hat{C}_1 \dots \hat{C}_{n-1} [\hat{A}, \hat{C}_n] \right) \hat{C}_{n+1}$$

$$+ \hat{C}_1 \hat{C}_2 \dots \hat{C}_n (\hat{A} \hat{C}_{n+1} - \hat{C}_{n+1} \hat{A})$$

$$= [\hat{A}, \hat{C}_1] \hat{C}_2 \dots \hat{C}_n \hat{C}_{n+1} + \hat{C}_1 [\hat{A}, \hat{C}_2] \hat{C}_3 \dots \hat{C}_n \hat{C}_{n+1}$$

$$\underbrace{+ \hat{C}_1 \dots \hat{C}_{n-1} [\hat{A}, \hat{C}_n] \hat{C}_{n+1} + \hat{C}_1 \hat{C}_2 \dots \hat{C}_n [\hat{A}, \hat{C}_{n+1}]}$$

$$\hat{C}_1 \dots \hat{C}_{n-1} (\hat{A} \hat{C}_n - \hat{C}_n \hat{A}) \hat{C}_{n+1} + \hat{C}_1 \dots \hat{C}_n (\hat{A} \hat{C}_{n+1} - \hat{C}_{n+1} \hat{A})$$

$$= \hat{C}_1 \dots \hat{C}_{n-1} \hat{A} \hat{C}_n \hat{C}_{n+1} - \hat{C}_1 \dots \cancel{\hat{C}_{n-1}} \hat{C}_n \hat{A} \hat{C}_{n+1}$$

$$+ \hat{C}_1 \dots \cancel{\hat{C}_n} \hat{A} \hat{C}_{n+1} - \hat{C}_1 \dots \hat{C}_n \hat{C}_{n+1} \hat{A}$$

$$= \hat{C}_1 \dots \hat{C}_{n-1} \hat{A} \hat{C}_n \hat{C}_{n+1} - \hat{C}_1 \dots \hat{C}_n \hat{C}_{n+1} \hat{A}$$

$$= \cancel{\hat{C}_1 \cdots \hat{C}_{n-1} \hat{C}_n \left(\hat{C}_1 \cdots \hat{C}_n \hat{A} \hat{C}_{n+1} - \hat{C}_1 \cdots \hat{C}_n \hat{C}_{n+1} \hat{A} \right.}$$

$$\cancel{= \hat{C}_1 \cdots \hat{C}_n [\hat{A}, \hat{C}_{n+1}]}$$

Was ich haben will:

$$[\hat{A}, \hat{C}_1 \hat{C}_2 \cdots \hat{C}_n \hat{C}_{n+1}]$$

$$= [\hat{A}, \hat{C}_1] \hat{C}_2 \cdots \hat{C}_n \hat{C}_{n+1} + \hat{C}_1 [\hat{A}, \hat{C}_2] \hat{C}_3 \cdots \hat{C}_n \hat{C}_{n+1}$$

$$+ \hat{C}_1 \cdots \hat{C}_n [\hat{A}, \hat{C}_{n+1}]$$